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Feedback Control of Bifurcation and Chaos in Dynamical Systems

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T.R. 93-74

Report Documentation Page			*Form Approved* *OMB No. 0704-0188*	

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1. REPORT DATE **1993**	2. REPORT TYPE	3. DATES COVERED **00-00-1993 to 00-00-1993**

4. TITLE AND SUBTITLE **Feedback Control of Bifurcation and Chaos in Dynamical Systems**	5a. CONTRACT NUMBER
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **University of Maryland,Systems Research Center,College Park,MD,20742**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
	11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution unlimited

13. SUPPLEMENTARY NOTES

14. ABSTRACT
see report

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**		**23**	

FEEDBACK CONTROL OF BIFURCATION AND CHAOS IN DYNAMICAL SYSTEMS

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ABSTRACT. Feedback control of bifurcation and chaos in nonlinear dynamical systems is discussed. The article summarizes some of the recent work in this area, including both theory and applications. Stabilization of period doubling bifurcations and of the associated route to chaos is considered. Open problems in bifurcation control are noted.

1 Introduction

The past two decades have witnessed a steadily increasing appreciation of nonlinear dynamics across a broad range of disciplines. Applications of bifurcation and chaos have appeared in many areas of science, engineering and the social sciences. Our main purpose in this article is to discuss the role which nonlinear dynamics has played in the cross-disciplinary field of automatic control. We also discuss in some detail the analysis and control of period doubling bifurcations, and application to control of the associated route to chaos. Thoughts on some open problems are given, emphasizing the needed interplay between nonlinear dynamics and control theory.

Bifurcations are qualitative changes in the phase portrait of a dynamical system that occur as a system parameter (a *bifurcation parameter*) is quasistatically varied. In this article we focus on dynamical systems depending on a scalar bifurcation parameter, and described by a set of ordinary differential or difference equations. Moreover, we suppose that for a nominal parameter range, a system of interest operates at a stable fixed point, i.e., at equilibrium. As the bifurcation parameter is varied, the fixed point can lose stability. This results in a bifurcation, in which new limit sets arise. These new limit sets are known as bifurcated solutions. Typical bifurcated solutions include fixed points and periodic

solutions in the vicinity of the nominal fixed point. The bifurcation may also entail the disappearance of the nominal fixed point, through a fold bifurcation. In this case, unless another limit set bifurcates from the nominal fixed point at the fold bifurcation, we would generally expect divergence of the system trajectory to another attractor. This would likely be associated with a loss of system performance, and possible system failure.

Bifurcations of fixed points and periodic orbits from a nominal fixed point are among the simplest in nonlinear dynamics. Elegant results for the analysis of these bifurcations are available. Limit sets emerging from such a bifurcation often undergo further bifurcations. To distinguish these bifurcations, the first bifurcation, which occurs along the nominal solution branch, is termed a primary bifurcation. A solution bifurcating off of the primary bifurcation branch is termed a secondary bifurcation. The successive bifurcations can become rather complex, and can lead to chaotic (or turbulent) behavior. Chaos is an irregular, seemingly random dynamic behavior displaying extreme sensitivity to initial conditions. Nearby initial conditions result, at least initially, in trajectories that diverge exponentially fast.

Among the areas discussed in this article, the control of bifurcation and chaos by feedback is our major focus. Our work on control of chaos is to a large extent an application of bifurcation control, which is a growing set of results for control of bifurcations of various types. Although control of nonlinear dynamics is in general an intractable subject, bifurcation control tends to focus on control problems which are localized in state space near the nominal fixed point. Large variations in parameter space are permitted in the theory. By controlling the location, amplitude and stability of primary bifurcations, it is often possible to achieve satisfactory performance over a wide range of parameters. Moreover, in some situations it may be advantageous to introduce new bifurcations from the nominal branch. These new bifurcated solutions may serve as signals of impending collapse ("stall"). Alternatively, they may be judiciously combined with existing dynamical features of the system to extend the operating region to parameter ranges which cannot be attained with linear control.

Our main motivation for the study of control of bifurcation and chaos relates to a performance vs. stability trade-off that appears in a variety of forms in various applications. It is often the case that significant improvement in performance is achieved by operation near the stability boundary. From the remarks above, such operation may well lead to bifurcation phenomena in the presence of small disturbances. Achieving increases in performance while maintaining an acceptable safety margin is an important current engineering challenge. An essential aspect of this challenge is the design of controllers which facilitate

operation of systems in nonlinear regimes with a negligible margin of stability. It is important to note that linearized models are not adequate for prediction or control of a system's response near the stability boundary.

It has not been our purpose in writing this article to perform a comprehensive literature review, nor to provide an exhaustive discussion of research on the control of bifurcations. We have attempted to balance discussion of our own work with a review of other research in the control of bifurcations. The reference list is incomplete, but an attempt was made to include significant contributions reflecting various views. With the availability of several recent reviews of control of chaos [18],[28],[64], we have not attempted to discuss this area in detail. We do, however, discuss how control of chaos can be achieved through control of bifurcations.

The article proceeds as follows. Section 2 contains a general discussion of the literature tying nonlinear dynamics to control theory. This literature is broadly classified as belonging either to the "nonlinear dynamics of control systems" category or to the "control of nonlinear dynamics category." In Section 3, we discuss some of our recent results on control of period doubling bifurcations. In Section 4 we briefly discuss the application of bifurcation control, including the results of Section 3, to the control of chaos. Concluding remarks are collected in Section 5.

> He who seeks for methods
> without having a definite problem in mind
> seeks for the most part in vain.
>
> – David Hilbert 1862-1943

2 Control and Nonlinear Dynamics

Nonlinear dynamic phenomena have always had some influence on control system design. For instance, the describing function method was developed to predict periodic solutions and their stability in controlled systems containing nonlinearities such as saturation and backlash; the knowledge gained could then be used to specify hardware tolerances [26]. In addition, control to delay the onset of aircraft stall resulted in stall of increased severity at a higher angle of attack [8].

More recently, advances in nonlinear dynamics have stimulated new research linking bifurcation, chaos and control. This research can be viewed as being divided into two categories. In the first category, efforts concentrated on studying nonlinear dynamics of control systems. That is, results of the *analysis* type were sought, but focusing on control systems. In the second category, efforts have been directed at control of bifurcation and chaos in dynamical systems. Thus, in this second stage

of research, emphasis has been placed on *design* techniques which result in prescribed nonlinear dynamics for controlled processes.

We note that some of the references, though of relevance to the subject of this article, do not fall under one of these categories. For instance, the paper [41] discusses the stimulation of bifurcation by feedback control as an aid in system identification.

2.1. Nonlinear Dynamics of Control Systems

Nonlinear dynamics of control systems has been studied from several viewpoints and with varying goals. In this subsection we discuss several such efforts.

Control theory includes a large body of results on stability and oscillation of nonlinear system models under parameter uncertainty. Many of these results are graphical in nature, and are expressed in terms of the system frequency response [48]. Criteria for bifurcation and chaos of general nonlinear systems have been obtained which are reminiscent of these classical control systems results. An advantage of these results is their inherent computational efficiency. Mees and Chua [49] gave a frequency domain version of the Andronov-Hopf Bifurcation Theorem, which includes a graphical test for stability of the bifurcated periodic solutions. Baillieul, Brockett and Washburn [13] gave sufficient conditions for chaotic behavior of a class of nonlinear systems, also expressed in the frequency domain. Genesio and Tesi [27] give frequency domain criteria for the approximate prediction of chaotic behavior. Their approach is based on the method of harmonic balance [26],[48].

Adaptive control techniques are algorithms for the continuous adjustment, or adaptation, of control laws as a means for coping with system uncertainty. Adaptive control schemes are inherently nonlinear. Even a linear uncertain system results in a nonlinear system when controlled adaptively. Thus, it is conceivable that an adaptively controlled linear system may exhibit oscillations and even chaotic behavior. This has been demonstrated by several researchers. Recent articles on this subject include Mareels and Bitmead [46],[47] and Salam and Bai [62].

Bifurcations induced by variation of control gains and system parameters in controlled systems have been considered by many authors (*e.g.*, [5], [16], [17], [19], [22], [33], [50], [56], [59]). Delchamps [22] has shown that the quantized linear feedback control of linear systems can result in chaotic behavior. Also, Chang and coworkers have systematically studied bifurcations and chaos induced by traditional linear control designs applied to practical, nonlinear systems. In [16], the bifurcation characteristics of nonlinear systems under conventional proportional-integral-derivative (PID) control are studied. It was found that the controlled system can exhibit a rich set of dynamic behaviors, including multiple equilibrium points, limit cycles, tori and strange attractors. In

[17] control of chaos is implemented in a system displaying intermittency. Global effects of controller saturation on system dynamics were investigated in [19].

2.2. Control of Nonlinear Dynamics

In this subsection we discuss a number of problem classes and approaches to the control of nonlinear dynamic phenomena. These problem classes share an emphasis on control of nonlinear dynamic phenomena. However, the specific objectives differ among the problem classes. The pertinence of any of these methods depends heavily on the envisioned application and performance requirements. The classes of problems discussed next are bifurcation control, control of chaos, and control of qualitative behavior.

Bifurcation control deals with using a control input to modify the bifurcation characteristics of a parametrized system. The control can be a static or dynamic feedback, or an open-loop control law. The objective of control can be stabilization and/or delay of a given bifurcation, reduction of the amplitude of bifurcated solutions, optimization of a performance index near bifurcation, re-shaping of a bifurcation diagram, or a combination of these.

Optimization and optimal control of bifurcation and branching are studied by, *e.g.*, Qin [58] and Doedel, Keller and Kernévez [23]. Open-loop control (as opposed to feedback) is studied by, *e.g.*, Baillieul [12], Colonius and Kliemann [21] and Tung and Shaw [71]. Baillieul uses a time-periodic forcing signal to delay an Andronov-Hopf bifurcation. Colonius and Kliemann use the control sets construct to determine bifurcations of the reachable set near an Andronov-Hopf bifurcation. Tung and Shaw consider the improvement in performance that can be obtained using open-loop control of a model of impact print hammer dynamics. An article by Antman and Adler [6] investigates the design of material properties to achieve a prescribed global buckling response. Cibrario and Lévine [20] considered the control of hysteretic bifurcation diagrams with application to thermal runaway of continuous stirred tank reactors. Henrich, Mingori and Monkewitz [32] use linear feedback to stabilize the nominal equilibrium point of a system undergoing pitchfork or Andronov-Hopf bifurcation.

Much of the research of the authors and coworkers in the area of bifurcation control relates to stabilization, or "softening," of bifurcations, and implications for improved system performance and robustness. Subsequent sections are devoted to recent results of the authors in this area, and to a discussion of applications. Thus it is not necessary to also summarize these results in this section. However we note that the need for control laws which soften (stabilize) a hard (unstable) bifurcation has been articulated by many in the past, in a variety of

contexts. For instance, stabilization of business cycles in the capitalist economy is considered by Foley [24] using a model which exhibits Andronov-Hopf bifurcation. Several other references at the end of this chapter also note the preference for soft bifurcations over hard bifurcations. This is closely related to the commonly employed terminology of "safe" vs. "dangerous" stability boundaries [63].

A recent flurry of activity in the *control of chaos* was sparked by the paper [54] of Ott, Grebogi and Yorke. Their method involves local stabilization of an unstable periodic orbit embedded in a chaotic attractor (see also [60]). The periodic orbit is selected to ensure a desired level of performance. Ergodicity of the chaotic attractor results in trajectories eventually entering a neighborhood of the stabilized periodic orbit. Thereafter, the system operates on the chosen periodic orbit. We mention one particular application of this method [30], since it illustrates the performance improvement that can be achieved using control of nonlinear dynamics. This is a study of control of a multimode laser well into its usually unstable regime.

The foregoing discussion of results on the control of chaos is of necessity very brief. In Section 4 we discuss the bifurcation control approach to the control of chaos. This approach entails considering a system over a parameter range which includes regular and chaotic regimes. The bifurcation sequence taking the system from regular behavior to chaotic behavior is controlled by imparting a sufficient degree of stability to a primary bifurcation in the sequence. A guiding theme in this approach is to maintain stability of bifurcated solution branches since they cannot give rise to secondary bifurcations unless they lose stability.

In the *control of qualitative behavior* of nonlinear systems, the goal is to determine a feedback control which transforms the phase portrait into a desired one. A typical problem is to determine a feedback which results in the introduction of a limit cycle in the neighborhood of an equilibrium point [66]-[68]. The method of entrainment and migration controls proposed by Jackson [37] may be viewed in this context. In this method, a control is sought to drive a system to follow a goal dynamics which is specified by a chosen dynamical model. The control of homoclinic orbits is considered by Bloch and Marsden [14]. They show that arbitrarily long residence times in the neighborhood of the homoclinic orbit can be achieved. They apply this result to the control of bursting phenomena in the near wall region of a turbulent boundary layer.

The angles of the boundary of stability
are always directed outside,
driving a wedge into the domain of instability.
This is apparently the consequence of a very general principle,
according to which everything good is fragile.

– Vladimir Arnold [7]

3 Bifurcation Control

In this section we describe our work on bifurcation control, emphasizing recent results. We begin by briefly summarizing the results of [1], [2].

3.1. Local Static State Feedback Stabilization

Our early work on bifurcation control [1], [2] focused on obtaining stabilizing feedback control laws for general one parameter families of nonlinear control systems

$$\dot{x} = f_\mu(x, u). \tag{1}$$

Here $x \in \mathbb{R}^n$ is the state vector, u is the scalar control, $\mu \in \mathbb{R}$ is the bifurcation parameter, and the map f_μ is smooth in x, u and μ. In [1], [2] it was assumed that Equation (1) with the control set to 0 undergoes either an unstable Andronov-Hopf bifurcation or an unstable stationary bifurcation from a nominal equilibrium point $x_0(\mu)$ at the critical parameter value $\mu = 0$. It was assumed that the equilibrium $x_0(\mu)$ exists and depends smoothly on μ in a neighborhood of $\mu = 0$.

The control laws derived in [1], [2] transform an unstable (i.e., subcritical) bifurcation into a stable (i.e., supercritical) bifurcation. These control laws were taken to be of the general form $u = u(x)$. These are known as *static state feedbacks*. Using Taylor series expansion of the vector field f_μ, smooth static state feedback control laws were designed rendering the assumed Andronov-Hopf bifurcation or stationary bifurcation locally attracting. Stability of the bifurcated solutions was measured using leading coefficients in Taylor expansions of the dominant characteristic exponents. The projection method of bifurcation analysis was employed [35]. Details are to be found in [1], [2].

Several applications of these results have been conducted. Among these are stabilization problems in tethered satellites [44], magnetic bearing systems [51], voltage dynamics in electric power systems [75], and compressor stall in gas turbine jet engines [10], [11], [45], [76]. In these applications, bifurcation control resulted in significant performance improvements. Successful experimental results have been reported in the area of compressor control for gas turbine jet engines [11].

We note that the results of [1], [2] though formulated for bifurcation problems, also provide solutions to feedback stabilization problems for critical nonlinear systems, i.e., systems with eigenvalues on the imaginary axis. The recent book [9] and the review paper [69] provide overviews of many nonlinear stabilization results, and the nonlinear stabilization area continues to be very active.

3.2. Dynamic Feedback in Bifurcation Control

Use of a static state feedback control law $u = u(x)$ has potential disadvantages in nonlinear control of systems exhibiting bifurcation behavior. In general, a static state feedback

$$u = u(x - x_0(\mu))$$

designed with reference to the nominal equilibrium path $x_0(\mu)$ of (1) will affect not only the stability of this equilibrium but also the location and stability of other equilibria. Now suppose that (1) is only an approximate model for the physical system of interest. Then the the nominal equilibrium branch will also be altered by the feedback. A main disadvantage of such an effect is the wasted control energy that is associated with the forced alteration of the system equilibrium structure. Other disadvantages are that system performance is often degraded by operating at an equilibrium which differs from the one at which the system is designed to operate.

For these reasons, we have developed bifurcation control laws for systems (1) which are *dynamic state feedback* control laws of a special form. Specifically, we have incorporated high pass filters known as *washout filters* into the structure of the allowed controllers. In this way, we guarantee preservation of all system equilibria even under model uncertainty. The discussion below follows our papers [42], [43], [73].

Washout filters are used commonly in control systems for power systems and aircraft. The main purpose of using these filters is to achieve equilibrium preservation in the presence of system uncertainties. A washout filter is a stable high pass filter with transfer function

$$G(s) = \frac{y(s)}{x(s)} = \frac{s}{(s+d)} \tag{2}$$

In the following, washout filters are incorporated into bifurcation control laws for (1). Specifically, in (1), for each system state variable x_i, $i = 1, \ldots, n$, introduce a washout filter governed by the dynamic equation

$$\dot{z}_i = x_i - d_i z_i \tag{3}$$

along with output equation

$$y_i = x_i - d_i z_i \tag{4}$$

Here, the d_i are positive parameters (this corresponds to using stable washout filters). Finally, we require that the control u depend only on the measured variables y, and that $u(y)$ satisfy $u(0) = 0$.

In this formulation, n washout filters, one for each system state, are present. In fact, the actual number of washout filters needed, and hence also the resulting increase in system order, can usually be taken less than n.

The advantages of using washout filters stem from the resulting properties of equilibrium preservation and automatic equilibrium (operating point) following. Indeed, since $u(0) = 0$, it is clear that y vanishes at steady state. Hence the x subvector of a closed loop equilibrium point (x, z) agrees exactly with the open loop equilibrium value of x. Also, since Eq. (4) can be written as

$$y_i = x_i - d_i z_i = (x_i - x_{0_i}(\mu)) - d_i(z_i - z_{0_i}(\mu)) \qquad (5)$$

the control function $u = u(y)$ is guaranteed to center at the correct operating point.

In [42], [43] stabilization of Andronov-Hopf bifurcations using washout filter-aided control laws is studied. Estimates for the tolerable degree of model uncertainty are obtained, and an application to a nonlinear aircraft model is given. In [29] the washout filter concept is used in the control of a periodic solution of a continuous-time system. Here, a judiciously chosen transfer function attenuates frequencies near that of the nominal periodic solution.

3.3. Control of Period Doubling Bifurcations

The three generic bifurcations for one parameter families of discrete-time maps are the fold bifurcation, the period doubling bifurcation, and the Neimark-Sacker-Moser bifurcation. In this section, we summarize results on the period doubling bifurcation as derived in [3], which may be consulted for details. In designing stabilizing control laws for the period doubling bifurcation, it is useful to have a framework for the analysis of these bifurcations and, specifically, their stability. The approach used here and the formulas we obtain are applicable directly to system (6), not requiring invariant manifold reduction or coordinate transformation. This approach is an instance of the projection method [35]. We note that Iooss and Joseph [35] perform an analysis of period doubling bifurcations for continuous-time systems using the projection method.

Period doubling bifurcations are most readily analyzed in a discrete-time setting. In discrete-time, the nominal periodic orbit (fixed point) is given whereas in the continuous-time setting it must be approximated as a waveform. Of course, to obtain the discrete-time model,

a device such as the Poincaré return map must be used, and this also involves approximation.

Next we give a simple derivation of the basic period doubling bifurcation result for an n-dimensional map, as well as an associated stability calculation. The projection method is employed in the derivation. This is followed by an application of the resulting expressions in the synthesis of stabilizing controllers.

Period Doubling Bifurcation Stability Analysis

Consider the system

$$x_{k+1} = F_\mu(x_k) \tag{6}$$

where k is an integer index, $x_k \in \mathbb{R}^n$, $\mu \in \mathbb{R}$ is the bifurcation parameter, and the mapping f_μ is sufficiently smooth in x and μ.

We proceed to derive a theorem which gives sufficient conditions for a period doubling bifurcation to occur for Equation (6). In the course of the derivation, we also obtain an explicit test for stability of the period doubled orbit. Not surprisingly, this test reduces to a standard calculation in the case of scalar maps [31]. The next hypothesis is invoked in the theorem.

(P) The map F_μ of Eq. (1) is sufficiently smooth and has a fixed point at $x = 0$ for $\mu = 0$. The linearization of (6) along the fixed point which is the continuous extension of the origin possesses an eigenvalue $\lambda_1(\mu)$ with $\lambda_1(0) = -1$ and $\lambda_1'(0) \neq 0$. All remaining eigenvalues of the linearization have magnitude less than unity.

Expanding the map F_μ in a Taylor series about $(x = 0, \mu = 0)$, we have

$$F_\mu(x) = A(\mu)x + Q(x, x) + C(x, x, x) + \dots$$

Here, $A(\mu)$ is a matrix, $Q(x, x)$ is a quadratic form generated by a symmetric bilinear form, and $C(x, x, x)$ is a cubic form generated by a symmetric trilinear form, and the dots indicate higher order terms in x and μ. (Reference [25] contains a discussion of multilinear forms which is relevant to this work.) We seek conditions under which a period-2 orbit bifurcates from $x = 0$ at $\mu = 0$.

Let $\ell :=$ the left eigenvector of $A(0)$ associated with the eigenvalue -1, and $r :=$ the right eigenvector of $A(0)$ associated with the eigenvalue -1.

Next, applying the recursion above to x_{k+1}, we have

$$\begin{aligned} x_{k+2} = {} & A^2(\mu)x_k + A(\mu)Q(x_k, x_k) + A(\mu)C(x_k, x_k, x_k) + \dots \\ & + Q(A(\mu)x_k + Q(x_k, x_k) + \dots, \ A(\mu)x_k + Q(x_k, x_k) + \dots) \\ & + C(A(\mu)x_k + Q(x_k, x_k) + \dots, \ \dots, \ \dots) + \dots \end{aligned} \tag{7}$$

A period-2 orbit x, if one exists, must therefore satisfy

$$
\begin{aligned}
0 =&(A^2(\mu) - I)x + A(\mu)Q(x, x) + A(\mu)C(x, x, x) + \dots \\
&+ Q(A(\mu)x + Q(x, x) + \dots, \ A(\mu)x + Q(x, x) + \dots) \\
&+ C(A(\mu)x + Q(x, x) + \dots, \ \dots, \ \dots) + \dots \\
=&:\tilde{A}(\mu)x + \tilde{Q}(x, x) + \tilde{C}(x, x, x) + \dots
\end{aligned}
\tag{8}
$$

where

$$
\begin{aligned}
\tilde{A}(\mu) &:= A^2(\mu) - I \\
\tilde{Q}(x, x) &:= A(0)Q(x, x) + Q(A(0)x, A(0)x) \\
\tilde{C}(x, x, x) &:= A(0)C(x, x, x) + 2Q(A(0)x, \ Q(x, x)) \\
&\quad + C(A(0)x, A(0)x, A(0)x)
\end{aligned}
\tag{9}
$$

Since

$$
A(0)r = -r, \quad \ell A(0) = -\ell
\tag{10}
$$

we have

$$
A^2(0)r = r, \quad \ell A^2(0) = \ell
\tag{11}
$$

which implies

$$
(A^2(0) - I)r = 0, \quad \ell(A^2(0) - I) = 0
\tag{12}
$$

Thus, $\tilde{A}(0)$ possesses a zero eigenvalue, which is seen also to be simple, by the Spectral Mapping Theorem. Also by this theorem we find that $\frac{d}{d\mu}\lambda_1(\tilde{A}(\mu))|_{\mu=0}$ is nonzero if $\frac{d}{d\mu}\lambda_1(A(\mu))|_{\mu=0} \neq 0$. This explains the presence of this latter condition in (P). The foregoing is a sketch of a proof for a theorem on period doubling bifurcation.

We have shown, by reducing the problem to one of standard stationary bifurcation analysis, that the system (6) possesses a nontrivial period doubled orbit $x(\epsilon)$ emanating from $x = 0$ for $\mu = \mu(\epsilon)$ near 0.

To determine the stability of the period doubled orbit, we obtain formulas for bifurcation stability coefficients. These are simply coefficients in the Taylor expansions in an amplitude parameter ϵ of the critical eigenvalue of the period doubled orbit. Let this eigenvalue be given by

$$
\beta(\epsilon) = \beta_1\epsilon + \beta_2\epsilon^2 + \dots
\tag{13}
$$

Then, using formulas obtained in [2] for stationary bifurcation stability coefficients, we find:

$$
\begin{aligned}
\beta_1 &= \ell\tilde{Q}(r, r) \\
&= \ell[A(0)Q(r, r) + Q(A(0)r, A(0)r)] \\
&= -\ell Q(r, r) + \ell Q(-r, -r) \\
&= -\ell Q(r, r) + \ell Q(r, r) \\
&= 0
\end{aligned}
\tag{14}
$$

As for β_2, we have:

$$\beta_2 = 2\ell[\tilde{C}(r,r,r) - 2\tilde{Q}(r, \tilde{A}^- \tilde{Q}(r,r))] \tag{15}$$

Here

$$\tilde{A}^- := (\tilde{A}^T \tilde{A} + \ell^T \ell)^{-1} \tilde{A}^T \tag{16}$$

This analysis shows that $\beta_1 = 0$ and that, *generically* $\beta_2 \neq 0$. Hence, we have that if $\lambda_1(0) = -1$, $\lambda_1'(0) \neq 0$, $\beta_2 \neq 0$, then there is a pitchfork bifurcation for the sped-up system, giving two period-2 orbits occurring either supercritically or subcritically. For the original system, this means there is a *single* period doubled orbit occurring either supercritically or subcritically. Whether the period doubled orbit is supercritical or subcritical is determined by the sign of β_2. The period doubled orbit is supercritical if $\beta_2 < 0$ but is subcritical if $\beta_2 > 0$. It is reassuring to note that specialization of this result to the case in which F_μ is a scalar map agrees with Theorem 3.5.1 in Guckenheimer and Holmes [31]. In fact, for scalar maps $\beta_2 = -2a$ where a is as given in [31, p. 158].

1. THEOREM. *(Period Doubling Bifurcation Theorem) If (P) holds, then a period doubled orbit bifurcates from the origin of (6) at $\mu = 0$. The period doubled orbit is supercritical and stable if $\beta_2 < 0$ but is subcritical and unstable if $\beta_2 > 0$.*

Stabilizing Controllers

We now consider the control of a period doubling bifurcation in the system (k is an integer)

$$x_{k+1} = f_\mu(x_k, u_k) \tag{17}$$

where $x_k \in \mathbb{R}^n$ is the state, u_k is a scalar control input, $\mu \in \mathbb{R}$ is the bifurcation parameter, and the mapping f_μ is sufficiently smooth in x, u and μ.

Suppose the zero-input version of (17) satisfies hypothesis (P). This leads to a period doubling bifurcation for the zero-input version of (17). Take the control u to be of the form

$$u(x_k) = x_k^T Q_u x_k + C_u(x_k, x_k, x_k)$$

where Q_u is a real symmetric $n \times n$ matrix and $C_u(x, x, x)$ is a cubic form generated by a scalar valued symmetric trilinear form $C_u(x, y, z)$. Note that $u(x_k)$ contains no constant terms or terms linear in x_k. A constant term would physically represent a continuous expenditure of control energy. The absence of a linear term in the control is intentional. This reduces the complexity of the calculations, and facilitates treatment of the bifurcated solution stabilization problem separately from that of

delaying the occurrence of the bifurcation to higher parameter values. Thus, this choice of structure of the control law reflects a two-stage control design philosophy in which linear terms in the control are used to modify the location of a bifurcation and nonlinear terms are used to modify its stability.

The following theorem summarizes stabilization results for period doubling bifurcations which are given in detail in [3]. These results have also been extended to incorporate washout filters in the control laws. Discrete-time versions of washout filters and washout filter-aided bifurcation control laws are also discussed in [3] and in other work in preparation.

2. THEOREM. *Suppose that hypothesis (P) holds for the zero-input version of (17). If the critical eigenvalue -1 is controllable for the associated linearized system, then there is a feedback $u_k(x_k)$, containing only third order terms in the components of x_k, that results in a locally stable bifurcated period-2 orbit for μ near 0. This feedback also stabilizes the origin for $\mu = 0$. If, on the other hand, the critical eigenvalue -1 is uncontrollable for the associated linearized system, then generically there is a feedback $u_k(x_k)$, containing only second order terms in the components of x_k, that results in a locally stable bifurcated period-2 orbit for μ near 0. This feedback also stabilizes the origin for $\mu = 0$.*

> Gaia is very chaotic,
> so if you reject chaos, you reject Gaia.
> – Ralph Abraham [15]

4 Control of Routes to Chaos

The bifurcation control techniques discussed in the foregoing have direct relevance for issues of control of chaotic behavior of dynamical systems.

There are many scenarios by which bifurcations can result in a chaotic invariant set. The current state of understanding differs considerably among the various known routes to chaos. These include the period doubling route, the Ruelle-Takens route, homoclinic bifurcation, intermittency and the devil's staircase. What is important about these scenarios from a control of chaos perspective is that chaos may be suppressed by controlling a bifurcation in a given route to chaos. The theme of our research in this area, as presented in [3],[73],[74], is to design feedback control laws which ensure a sufficient degree of stability for a primary bifurcation in such a scenario. We have successfully addressed the homoclinic and period doubling routes to chaos using this approach. The design presented in the foregoing section is an important component of our approach for controlling the period doubling route to chaos.

We note that the control laws of [3], [73], [74] leave unaffected the locations of the nominal equilibrium points, some benefits of which have been discussed earlier. The effectiveness of the control laws in achieving stabilization and equilibrium preservation persists even in the presence of model uncertainty.

It is the faith that it is the privilege of man
to learn to understand, and that this is his mission.

– Vannevar Bush 1967

5   Concluding Remarks

In the last decade, there has been a marked change in the way engineers and scientists interact with nonlinear dynamics. Whereas ten years ago nonlinear dynamics to most of us was somewhat of a novelty, it has become an indispensable part of our toolkit. The book [39] provides excellent examples of how engineers and scientists are using nonlinear dynamics concepts in understanding and controlling the behavior of real-world systems. Notably, the contributors to [39] were brought together through the efforts of an industrial organization. Engineers and scientists, including practitioners and researchers, are posing difficult unsolved problems to the applied mathematicians. This said, we would like to mention three problem areas related to the subject of this paper.

The first problem is to extend the results presented here on stabilization of the period doubling route to chaos to systems described by differential equations. The solution would need to include a method for approximating the nominal periodic solution of interest. With knowledge of the family of periodic solutions, stability coefficients may be calculated as in [35].

The second problem concerns the control of resonance (as opposed to bifurcation) effects in nonlinear systems. Systems can exhibit undesirable dynamic behavior, such as sharp but continuous increases in amplitude, as a parameter is varied. These nonlinear effects may be caused by internal or other resonances.

Finally, we mention the problem of controlling a dynamical system near a fold bifurcation, without altering the nominal equilibrium branch. Controlling the system so as to stabilize the equilibrium up to the fold is generally possible. However, what is a good design if the model is uncertain? The design should detect proximity to the fold even without access to an accurate model. Virgin [72] and others have studied related problems. An even more challenging problem is to design a control law allowing operation past the fold bifurcation. The motivation behind this problem is not to deliberately operate a system in such a mode. The risk of collapse would be too great. Rather, such a design would increase the system's margin of stability significantly be-

yond what is currently considered possible. The result would be allowing system operation closer to the fold bifurcation than would otherwise be possible. How can such a control law be found? Certainly bifurcations would need to be introduced in the vicinity of the fold bifurcation, resulting in stable limit sets of some type past the fold bifurcation. The limit sets could possibly be periodic or chaotic attractors. Degenerate bifurcations and spontaneous generation of chaotic attractors, possibly through intermittency, are ideas that may prove useful in this context. The Shoshitaishvili Reduction Theorem [65], [7, pp. 265-267] may play a role in addressing this issue. However, it appears likely that much of the needed theory is not yet in existence.

Acknowledgments

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